Ensemble phase-averaged equations for bubbly flows

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A system of averaged equations describing the motion of a mixture of spherical compressible bubbles in an inviscid liquid is derived by the ensemble-averaging method introduced in an earlier paper [Zhang and Prosperetti, J. Fluid Mech. **267**, 185 (1994)]. The averaging procedure introduces new terms in the equations, among which a contribution to the liquid stress tensor is of special interest. An extension of the well-known Rayleigh–Plesset equation to the case of bubbles interacting with the flow is also found. The general system of equations is closed in a rigorous manner in the dilute limit generalizing and correcting earlier averaged equations models. The results are illustrated by considering the problem of linear pressure wave propagation in a nonuniform bubbly liquid. Gradients of the bubble concentration are shown to dampen or amplify the wave strength.

I. INTRODUCTION

This paper is a continuation of our recent work,[1] (see also Ref. 2) where a phasic ensemble-average approach to the modeling of inertial effects in a two-phase mixture of rigid spheres suspended in an incompressible fluid was developed and demonstrated. Here, we extend that technique to the case in which the radius of the suspended particles responds to the pressure field in the surrounding continuous phase. Thus the present results are directly relevant to the modeling of the flow of liquids containing bubbles sufficiently large that inertial effects dominate their motion in the liquid, and at the same time sufficiently small that deviations from the spherical shape can be neglected. For air bubbles in water these constraints are approximately satisfied for diameters of the order of 1 mm, for which the Reynolds number based on the terminal rise velocity is of the order of 100 the hydrostatic effect less than 10%, and the dynamical pressure at the interface less than 1% the surface-tension overpressure.[3]

Beyond this direct application, the present paper has other intents. In the first place, the radius is no different in principle from other degrees of freedom of the suspended particles, such as shape or orientation. The technique developed here will show how these additional particle attributes can be incorporated in the scheme of Ref. 1. Second, it will be seen that our results are related to, or contain as special cases, earlier ones available in the literature. They are therefore useful to judge their validity, explore their limitations, and generally provide a rigorous unifying framework.

The advantages of the present phasic ensemble averaging have been discussed in Ref. 1. Briefly, the phasic averaging leads directly to a two-fluid formulation of the type generally used in the engineering literature. The disperse-phase equations are derived without recourse to the artifices usually needed when dealing with degenerate equations of state, such as negligible density. Finally, the closure problem presents itself in the form of *computable* quantities, for which approximate methods or direct numerical simulation are suitable. This point is illustrated by considering explicitly the closure problem in the dilute limit. In particular, no *ad hoc* techniques such as the so-called "cell model" prove necessary for this calculation.

The results are illustrated by considering the propagation of linear pressure waves in a bubbly mixture including nonuniformities of the bubble distribution.

A comparison with previous work is given in Sec. IX.

II. AVERAGING RELATIONS

A detailed exposition of the averaging techniques and theorems used in this paper is given in Ref. 1. Here, we summarize the relevant results omitting the proofs.

Consider an ensemble of two-phase flows in which each realization contains N particles (or bubbles) arranged in a time-dependent configuration $\mathscr{C}^N$. We use this word and symbol in a technical sense as short-hand for the set of position vectors of the particle centers, $\mathbf{y}^\alpha$, $\alpha=1,2,\ldots,N$, their radii a^α, their translational velocities $\mathbf{w}^\alpha$, and their radial velocities $\dot{a}^\alpha$. For brevity we will refer to $\mathbf{q}^\alpha=(\mathbf{y}^\alpha,a^\alpha)$ as the (generalized) positions, and to $\mathbf{p}^\alpha=(\mathbf{w}^\alpha,\dot{a}^\alpha)$ as the (generalized) velocities. For each member of the ensemble, with suitable initial conditions and dynamical equations for the particles and the continuous phase, and conditions "at infinity" for the latter, the microscopic, exact problem is uniquely specified.

Let $P(N;t)$ be the probability of a specific configuration $\mathscr{C}^N$ at time t. We take all the bubbles to have the same equilibrium radius so that they are indistinguishable. (This does not prevent of course bubbles with the same position and velocity in different realizations from having different radii.) Therefore, $N!$ different arrangements of the N bubbles correspond to each physically distinguishable state of the system. A convenient normalization is then (see Ref. 4)

$$\int P(N;t)d\mathscr{C}^N=N!, \tag{1}$$

where $d\mathscr{C}^N=d\mathbf{p}^{(1)}\,d\mathbf{q}^{(1)}\cdots d\mathbf{p}^{(N)}\,d\mathbf{q}^{(N)}$ is the volume element in the phase space of the system. The reduced probability distribution in which the configuration of K particles is prescribed is obtain by integration

$$P(K;t)=\frac{1}{(N-K)!}\int P(N;t)d\mathscr{C}^{N-K}, \tag{2}$$

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and satisfies the normalization condition

$$\int P(K;t)d\mathscr{C}^K=\frac{N!}{(N-K)!}. \tag{3}$$

The fraction in the right-hand side is the number of ways in which K objects can be chosen from a group of N.

Let $\chi_C(\mathbf{x};N)$ and $\chi_D(\mathbf{x};N)$ be the indicator functions of the continuous and disperse phases so that, for example, $\chi_C=1$ when, with the particles in the configuration $\mathscr{C}^N$, the point $\mathbf{x}$ is in the continuous phase, and $\chi_C=0$ otherwise. Note that these are geometrical entities that depend on time only indirectly through the time evolution of the configuration. The volume fractions of the two phases are defined by

$$\beta_{C,D}=\frac{1}{N!}\int d\mathscr{C}^N\, P(N;t)\chi_{C,D}(\mathbf{x};N). \tag{4}$$

The particle boundary is assumed to have zero measure so that $\chi_C+\chi_D=1$ and, from Eq. (1),

$$\beta_C+\beta_D=1. \tag{5}$$

At this point we specialize the treatment to spheres by assuming the following specific form for χ_D:

$$\chi_D(\mathbf{x};N)=\sum_{\alpha=1}^{N} H(a^\alpha-|\mathbf{x}-\mathbf{y}^\alpha|)=1-\chi_C(\mathbf{x};N), \tag{6}$$

with H the Heaviside distribution. By the same method followed in Ref. 1 it is easy to show that, with this expression, definition (4) gives

$$\beta_D(\mathbf{x},t)=\int da\int_{|\mathbf{x}-\mathbf{y}|\leq a} d^3y\int d^4p\; P(1;t), \tag{7}$$

where $P(1;t)\equiv P(\mathbf{y},a,\mathbf{p};t)\equiv P(\mathbf{y},a,\mathbf{w},\dot{a};t)$ and $\int d^4p=\int d^3w\int d\dot{a}$. For future use it is convenient to introduce two auxiliary quantities related to this result, the number density n_a of particles with radius a,

$$n_a(\mathbf{x},a,t)=\int d^4p\; P(\mathbf{x},a,\mathbf{p};t), \tag{8}$$

and the contribution β_a that they give to the disperse-phase volume fraction

$$\beta_a(\mathbf{x},a,t)=\int_{|\mathbf{x}-\mathbf{y}|\leq a} d^3y\int d^4p\; P(1;t)$$
$$=\int_{|\mathbf{x}-\mathbf{y}|\leq a} d^3y\; n_a(\mathbf{y},a,t). \tag{9}$$

Clearly,

$$\beta_D(\mathbf{x},t)=\int da\; \beta_a(\mathbf{x},a,t). \tag{10}$$

It will be noted that, in the presence of gradients of the particle distribution, the relation $\beta_a=n_a v_a$, with $v_a=\frac{4}{3}\pi a^3$, is not strictly valid. The total particle number density is given by

$$n(\mathbf{x},t)=\int da\; n_a(\mathbf{x},a,t). \tag{11}$$

The phase ensemble average for a field $f_{C,D}(\mathbf{x},t;N)$ pertaining to the continuous or disperse phase is defined by averaging over all the configurations such that the point $\mathbf{x}$ is in the appropriate phase

$$\langle f_{C,D}\rangle(\mathbf{x},t)=\frac{1}{\beta_{C,D}N!}\int d\mathscr{C}^N\, f_{C,D}(\mathbf{x},t;N)\chi_{C,D}(\mathbf{x};N)P(N;t). \tag{12}$$

A similar definition is used for the conditional averages, i.e., averages over all configurations such that some particles occupy a specified configuration. Specifically, the conditional averages with one particle held fixed are defined by

$$\langle f_{C,D}\rangle_1(\mathbf{x},t|1)=\frac{1}{(N-1)!\beta^1_{C,D}}\int d\mathscr{C}^{N-1}\, \chi_{C,D}(\mathbf{x};N)f_{C,D}(\mathbf{x},t;N)\times P(N-1|1;t), \tag{13}$$

where the conditional volume fractions are given by

$$\beta^1_{C,D}(\mathbf{x},t|1)=\frac{1}{(N-1)!}\int d\mathscr{C}^{N-1}\times P(N-1|1;t)\chi_{C,D}(\mathbf{x};N). \tag{14}$$

Here, we have introduced the conditional probability $P(N-K|K;t)$ defined by

$$P(N;t)=P(K;t)P(N-K;t), \tag{15}$$

and subject to the normalization

$$\int P(N-K|K;t)d\mathscr{C}^{N-K}=(N-K)!, \tag{16}$$

consequence of Eqs. (15) and (3).

A. Continuous-phase averaging relations

By the same method used in Ref. 1 it is readily shown that

$$\nabla(\beta_C\langle f_C\rangle)=\beta_C\langle\nabla f_C\rangle+\int da\int_{|\mathbf{x}-\mathbf{y}|=a} dS_y\,\mathbf{n}\times\int d^4p\; P(1;t)\langle f_C\rangle_1(\mathbf{x},t|1). \tag{17}$$

With $f_C=1$ this result gives, by Eqs. (5) and (10),

$$\nabla\beta_C=\int da\int_{|\mathbf{x}-\mathbf{y}|=a} dS_y\,\mathbf{n}\int d^4p\; P(1;t), \tag{18}$$

so that Eq. (17) may be equivalently written

$$\nabla\langle f_C\rangle=\langle\nabla f_C\rangle+\frac{1}{\beta_C}\int da\int_{|\mathbf{x}-\mathbf{y}|=a} dS_y\,\mathbf{n}\int d^4p\; P(1;t)\times[\langle f_C\rangle_1(\mathbf{x},t|1)-\langle f_C\rangle(\mathbf{x},t)]. \tag{19}$$

This relation shows that, in the present approach, averaging and differentiation do not commute as in the methods where averaging is carried out irrespective of the phase occupying the position $\mathbf{x}$ (see, e.g., Ref. 5). The counterpart for this technical inconvenience is a much greater flexibility in the type of quantities that can be averaged as they do not necessarily have to be defined in both phases.

The same procedure of Ref. 1 gives, for the time derivative,

$$\frac{\partial}{\partial t}(\beta_C\langle f_C\rangle)=\beta_C\left\langle\frac{\partial f_C}{\partial t}\right\rangle-\int da\int_{|\mathbf{x}-\mathbf{y}|=a}dS_y \times\int d^4p(\dot{a}+\mathbf{w}\cdot\mathbf{n})P(1;t)\langle f_C\rangle_1. \quad (20)$$

The quantity $(\dot{a}+\mathbf{w}\cdot\mathbf{n})$ is the normal velocity of the particle surface and represents the local rate of change of the disperse-phase volume fraction. For $f_C=1$ this relation reduces to

$$\frac{\partial\beta_C}{\partial t}=-\int da\int_{|\mathbf{x}-\mathbf{y}|=a}dS_y\int d^4p(\dot{a}+\mathbf{w}\cdot\mathbf{n})P(1;t), \quad (21)$$

so that Eq. (20) may be equivalently written

$$\frac{\partial\langle f_C\rangle}{\partial t}=\left\langle\frac{\partial f_C}{\partial t}\right\rangle-\frac{1}{\beta_C}\int da\int_{|\mathbf{x}-\mathbf{y}|=a}dS_y \times\int d^4p(\dot{a}+\mathbf{w}\cdot\mathbf{n})P(1;t)[\langle f_C\rangle_1-\langle f_C\rangle]. \quad (22)$$

Combining Eqs. (19) and (22), one finds

$$\frac{\partial(\beta_C\langle f_C\rangle)}{\partial t}+\nabla\cdot(\langle\beta_C\langle f_C\mathbf{u}_C\rangle)=\beta_C\left\langle\frac{\partial f_C}{\partial t}+\nabla\cdot(f_C\mathbf{u}_C)\right\rangle-\int da\int_{|\mathbf{x}-\mathbf{y}|=a}dS_y\int d^4p\ P(1;t)\times[(\dot{a}+\mathbf{n}\cdot\mathbf{w})\langle f_C\rangle_1-\mathbf{n}\cdot\langle\mathbf{u}_C f_C\rangle_1]. \quad (23)$$

The surface integral is over all the particles touching $\mathbf{x}$. For all such particles the velocity field satisfies the kinematic boundary condition

$$\dot{a}+\mathbf{n}\cdot\mathbf{w}=\mathbf{n}\cdot\mathbf{u}_C(\mathbf{x},t|N), \quad (24)$$

and, since $(\dot{a}+\mathbf{n}\cdot\mathbf{w})\langle f_C\rangle_1=\langle(\dot{a}+\mathbf{n}\cdot\mathbf{w})f_C\rangle_1$, and $\mathbf{n}\cdot\langle f_C\mathbf{u}_C\rangle_1=\langle\mathbf{n}\cdot\mathbf{u}_C f_C\rangle_1$, the integral term in Eq. (23) vanishes identically so that

$$\frac{\partial(\beta_C\langle f_C\rangle)}{\partial t}+\nabla\cdot(\beta_C\langle f_C\mathbf{u}_C\rangle)=\beta_C\left\langle\frac{\partial f_C}{\partial t}+\nabla\cdot(f_C\mathbf{u}_C)\right\rangle. \quad (25)$$

Thus, though not commuting, averaging and convective differentiation satisfy a simple relation that will play a central role in the derivation of the averaged equations of the following section.

B. Disperse-phase averaging relations

By use of the explicit representation (6) of the disperse-phase indicator function and of the indistinguishability of the particles, definition (12) of the disperse-phase average may be written

$$\langle f_D\rangle(\mathbf{x},t)=\frac{1}{\beta_D}\int da\int_{|\mathbf{y}-\mathbf{x}|\leq a}d^3y\int d^4p\langle f_D^{(1)}\rangle_1 P(1;t), \quad (26)$$

where $f_D^{(1)}(\mathbf{x},t;N)$ is the value of f_D inside particle 1 when the particles are in the configuration $\mathscr{C}^N$. If this quantity does not depend explicitly on the configuration of the other particles, this relation simplifies to

$$\langle f_D\rangle(\mathbf{x},t)=\frac{1}{\beta_D}\int da\int_{|\mathbf{y}-\mathbf{x}|\leq a}d^3y\int d^4p\ f_D^{(1)}(\mathbf{x},t)P(1;t). \quad (27)$$

Transport equations analogous to Eq. (25) can be developed for these averages but they will not be needed in the following.

In dealing with the particle phase a different kind of average is necessary for quantities $g^{(\alpha)}(t;N)$ pertaining to the particle as a whole. Examples are the center-of-mass velocity, momentum, radius, and others. For such quantities we define the ensemble average over all the configurations such that the center of one particle having radius a is at $\mathbf{y}$ at time t by

$$\bar{g}^a(\mathbf{y},a,t)=\frac{1}{n_a}\frac{1}{(N-1)!}\int d^4p\int d\mathscr{C}^{N-1}\ P(N;t)\times g^{(1)}(t;N). \quad (28)$$

If the quantity $g^{(1)}$ does not depend explicitly on the configuration of the other particles, this definition reduces to

$$\bar{g}^a(\mathbf{x},a,t)=\frac{1}{n_a}\int d^4p\ P(1;t)g^{(1)}(\mathbf{x},a,\mathbf{p};t). \quad (29)$$

Note that, according to this definition, $\bar{a}^a=a$, although $\bar{\dot{a}}^a\neq\dot{a}$. The quantity $\bar{g}^a$ represents the contribution of the particles of radius a to the average. The average over all particle radii is given by

$$\bar{g}(\mathbf{x},t)=\frac{1}{n}\int da\ n_a(\mathbf{x},a,t)\bar{g}^a(\mathbf{x},a,t). \quad (30)$$

It is clear from this definition that $\bar{a}\neq a$ in general.

As in Ref. 1, it is easy to show for such particle averaged quantities that

$$\frac{\partial}{\partial t}(n_a\bar{g}^a)+\nabla\cdot(n_a\overline{\mathbf{w}g}^a)+\frac{\partial}{\partial a}(n_a\overline{\dot{a}g}^a)=n_a\overline{\frac{\partial g^{(1)}}{\partial t}}^a, \quad (31)$$

where the time derivative in the right-hand side is taken following the evolution of the system:

$$\frac{\partial g^{(1)}}{\partial t}=\frac{\hat{\partial}g^{(1)}}{\hat{\partial}t}+\sum_{\alpha=1}^{N}[\mathbf{p}^\alpha\cdot\nabla_{q^\alpha}g^{(1)}+\dot{\mathbf{p}}^\alpha\cdot\nabla_{p^\alpha}g^{(1)}], \quad (32)$$

with $\hat{\partial}/\hat{\partial}t$ the derivative with respect to the explicit time dependence of $g^{(1)}$, i.e., for fixed $\mathbf{x}$ and fixed configuration $\mathscr{C}^N$. A particular case of Eq. (31) that will arise later is

$$\frac{\partial}{\partial t}(n_a\bar{\dot{a}}^a)+\nabla\cdot(n_a\overline{\mathbf{w}\dot{a}}^a)+\frac{\partial}{\partial a}(n_a\overline{\dot{a}^2}^a)=n_a\bar{\ddot{a}}^a. \quad (33)$$

Since the field $g^{(1)}$ may be regarded as defined at the particle centers, the result (31) is the same as in the kinetic theory of gases, where the finite size of the molecules is disregarded.

We shall also use the form of the transport theorem (31) obtained by integration over a from 0 to infinity. By use of definition (30) one readily finds

$$\frac{\partial}{\partial t}(n\bar{g})+\nabla\cdot(n\overline{\mathbf{w}g})=n\,\overline{\frac{\partial g^{(1)}}{\partial t}}. \quad (34)$$

We now use these relations to derive the equations of motion of the phases.

III. AVERAGE EQUATIONS

With the neglect of viscous effects and compressibility, the equations of motion of the continuous phase are

$$\nabla\cdot\mathbf{u}_C=0, \quad (35)$$

$$\frac{\partial\mathbf{u}_C}{\partial t}+\nabla\cdot(\mathbf{u}_C\mathbf{u}_C)=-\frac{1}{\rho_C}\nabla p_C+\mathbf{g}. \quad (36)$$

Upon taking $f_C=1$ in Eq. (25) and using the continuity equation (35) we find the averaged continuity equation for the continuous phase as

$$\frac{\partial\beta_C}{\partial t}+\nabla\cdot(\beta_C\langle\mathbf{u}_C\rangle)=0. \quad (37)$$

Similarly, upon taking $f_C=\mathbf{u}_C$, we have from Eqs. (25) and (36)

$$\rho_C\frac{\partial}{\partial t}(\beta_C\langle\mathbf{u}_C\rangle)+\rho_C\nabla\cdot(\beta_C\langle\mathbf{u}_C\rangle\langle\mathbf{u}_C\rangle)$$
$$=-\beta_C\nabla\langle p_C\rangle+\beta_C\mathbf{A}_C(\mathbf{x},t)+\nabla\cdot(\beta_C\mathbf{M}_C)+\beta_C\rho_C\mathbf{g}, \quad (38)$$

where we have introduced the "kinematic" Reynolds stress tensor

$$\mathbf{M}_C=\langle\mathbf{u}_C\rangle\langle\mathbf{u}_C\rangle-\langle\mathbf{u}_C\mathbf{u}_C\rangle=-\langle(\mathbf{u}_C-\langle\mathbf{u}_C\rangle)(\mathbf{u}_C-\langle\mathbf{u}_C\rangle)\rangle, \quad (39)$$

and we have set

$$\mathbf{A}_C(\mathbf{x},t)=\nabla\langle p_C\rangle-\langle\nabla p_C\rangle, \quad (40)$$

or, from Eq. (19),

$$\beta_C\mathbf{A}_C(\mathbf{x},t)=\int da\int_{|\mathbf{x}-\mathbf{y}|=a}dS_y\,\mathbf{n}\int d^4p\;P(1;t)$$
$$\times[\langle p_C\rangle_1(\mathbf{x},t|1)-\langle p_C\rangle(\mathbf{x},t)]. \quad (41)$$

Equation (38) has been written in terms of the gradient of the average pressure as in the standard two-fluid models. This choice introduces the quantity $\mathbf{A}_C$ which, with $\mathbf{M}_C$, contains all the local phase momentum interactions. Explicit expressions for these quantities valid in the dilute limit are given later.

Upon setting $g^{(1)}=1$ in Eq. (31) we find the conservation equation for the number density of bubbles with radius a

$$\frac{\partial n_a}{\partial t}+\nabla\cdot(n_a\bar{\mathbf{w}}^a)+\frac{\partial}{\partial a}(n_a\bar{\dot{a}}^a)=0. \quad (42)$$

The conservation relation for the total number density irrespective of the value of a follows from Eq. (34) and is

$$\frac{\partial n}{\partial t}+\nabla\cdot(n\bar{\mathbf{w}})=0. \quad (43)$$

By using this result the general transport theorem (34) gives

$$\frac{\partial\bar{g}}{\partial t}+\bar{\mathbf{w}}\cdot\nabla\bar{g}-\frac{1}{n}\nabla\cdot[n(\bar{g}\bar{\mathbf{w}}-\overline{g\mathbf{w}})]=\overline{\frac{\partial g^{(1)}}{\partial t}}. \quad (44)$$

In particular, upon taking $g^{(1)}=a$, this relation gives

$$\bar{\dot{a}}=\frac{\partial\bar{a}}{\partial t}+\bar{\mathbf{w}}\cdot\nabla\bar{a}-\frac{1}{n}\nabla\cdot[n(\bar{a}\bar{\mathbf{w}}-\overline{a\mathbf{w}})]. \quad (45)$$

Similarly,

$$\bar{\ddot{a}}=\frac{\partial\bar{\dot{a}}}{\partial t}+\bar{\mathbf{w}}\cdot\nabla\bar{\dot{a}}-\frac{1}{n}\nabla\cdot[n(\bar{\dot{a}}\bar{\mathbf{w}}-\overline{\dot{a}\mathbf{w}})]. \quad (46)$$

If we were to follow the same procedure as for the continuous phase to derive the averaged momentum equation for the disperse phase, the result would involve the constitutive relations of that phase's material. This is undesirable for situations such as the present one, where the approximation $\rho_D\simeq 0$ may be quite accurate, or such as that in Ref. 1, where the particles are modeled as rigid. In order to avoid this problem, we shall average directly the equation of motion of the individual bubbles. For simplicity we neglect the gas mass and therefore the net force on the bubbles must vanish so that

$$0=-\int_{|\mathbf{x}-\mathbf{y}|=a}dS_y\,\mathbf{n}p_C(\mathbf{y},t;N). \quad (47)$$

The integral is over the surface of the bubble centered at $\mathbf{x}$. (In principle, collisions are included in this expression through their effect on p_C.) Upon averaging this relation according to Eq. (28) we find

$$0=-\int_{|\mathbf{x}-\mathbf{y}|=a}dS_y\,\mathbf{n}\int d^4p\langle p_C\rangle_1(\mathbf{y},t|1)P(\mathbf{x},a,\mathbf{p};t), \quad (48)$$

or, with the definition

$$\mathbf{A}_a=\nabla\langle p_C\rangle-\frac{1}{\beta_a}\int_{|\mathbf{x}-\mathbf{y}|=a}dS_y\,\mathbf{n}\int d^4p\;P(\mathbf{x},a,\mathbf{p};t)$$
$$\times\langle p_C\rangle_1(\mathbf{y},t|1), \quad (49)$$

$$0=-\nabla\langle p_C\rangle+\mathbf{A}_a. \quad (50)$$

Upon multiplying by β_a and integrating over a from 0 to infinity this relation becomes

$$0=-\beta_D\nabla\langle p_C\rangle+\beta_D\mathbf{A}_D, \quad (51)$$

where we introduce the definition

$$\beta_D \mathbf{A}_D = \int da\ \beta_a \mathbf{A}_a. \tag{52}$$

We assume that the gas pressure inside each bubble is spatially uniform (see, e.g., Refs. 6 and 7). If p_B denotes this quantity for the bubble centered at $\mathbf{x}$, the normal stress balance at the bubble surface is expressed by

$$p_B = \frac{2\sigma}{a} + p_C, \tag{53}$$

where σ is the surface tension coefficient. This condition cannot be imposed exactly without allowing the bubbles to deform. Therefore, here, we impose it in a surface-averaged form as

$$p_B = \frac{2\sigma}{a} + \frac{1}{4\pi a^2} \int_{|\mathbf{y}-\mathbf{x}|=a} dS_y\ p_C(\mathbf{y},t;N). \tag{54}$$

In the simplest case the internal pressure may be taken to be a function of the bubble radius, e.g., according to a polytropic relation, or may be taken constant to simulate a condensible vapor. Better alternatives exist (see, e.g., Ref. 8 and Sec. VIII below). For the present purposes however there is no need to state explicitly how p_B is to be calculated. Upon substituting Eq. (54) into the definition (29) of averaging and defining

$$Q_a(\mathbf{x},a,t) = \frac{1}{4\pi a^2 n_a} \int_{|\mathbf{y}-\mathbf{x}|=a} dS_y \int d^4p\ P(1;t) \times \langle p_C \rangle_1(\mathbf{y},t|1) - \langle p_C \rangle(\mathbf{x},t), \tag{55}$$

we find

$$\overline{p_B}^{a}(\mathbf{x},a,t) = \frac{2\sigma}{a} + \langle p_C \rangle(\mathbf{x},t) + Q_a(\mathbf{x},a,t). \tag{56}$$

It will be seen below that, at least in the dilute case, this relation bears a strong similarity to the Rayleigh–Plesset equation.

Compressibility effects of the continuous phase would readily be incorporated in the previous developments by replacing (35) by the standard continuity equation for a compressible fluid. The result would then be

$$\frac{\partial(\beta_C \langle \rho_C \rangle)}{\partial t} + \nabla \cdot (\beta_C \langle \rho_C \mathbf{u}_C \rangle) = 0, \tag{57}$$

in place of Eq. (37). Further progress would require the possibility to express $\langle \rho_C \mathbf{u}_C \rangle$ in terms of known quantities. The incorporation of viscosity is a more complex matter discussed in Ref. 9.

IV. SMALL-PARTICLE APPROXIMATION

The equations derived in the preceding sections contain several terms involving integration over spheres with a radius equal to the bubble radius a. When the macroscopic quantities vary slowly over this scale, these integrals may be approximated by Taylor series expansions. The detailed calculations are given in Ref. 1. Here, we simply note the pertinent results.

The volume fraction β_a contributed by the bubbles of radius a to the disperse-phase volume fraction is approximately given by

$$\beta_a(\mathbf{x},a,t) = v_a n_a(\mathbf{x},a,t)[1 + O(a^2/L^2)], \tag{58}$$

where L is the characteristic length for the variation of macroscopic quantities and $v_a = \frac{4}{3}\pi a^3$. Similarly, the contribution $\langle \mathbf{u}_a \rangle$ given by the bubbles of radius a to the average velocity of the disperse phase [i.e., defined according to Eq. (26) rather than Eq. (28)] can be shown to be

$$\langle \mathbf{u}_a \rangle(\mathbf{x},a,t) = \tilde{\mathbf{w}}^a(\mathbf{x},a,t) + O(a^2/L^2). \tag{59}$$

The average velocity of the disperse phase is given in terms of $\langle \mathbf{u}_a \rangle$ by a relation analogous to Eq. (52), i.e.,

$$\beta_D \langle \mathbf{u}_D \rangle = \int da\ \beta_a \langle \mathbf{u}_a \rangle. \tag{60}$$

To the same accuracy, the quantity $\mathbf{A}_a$ introduced in Eq. (49) is given by (Ref. 1)

$$\beta_a \mathbf{A}_a(\mathbf{x},a,t) = -\int_{|\mathbf{x}-\mathbf{y}|=a} dS_y\ \mathbf{n} \int d^4p\ P(\mathbf{x},a,\mathbf{p};t) \times [\langle p_C \rangle_1(\mathbf{y},t|1) - \langle p_C \rangle(\mathbf{y},t)]. \tag{61}$$

Furthermore, by the same method as in Ref. 1, it can be shown that, with an error of order $\beta_a a^2/L^2$,

$$\beta_C \mathbf{A}_C = -\beta_D \mathbf{A}_D + \nabla \cdot (\beta_D \mathbf{T}_C), \tag{62}$$

where $\mathbf{A}_D$ is defined by Eq. (52) and

$$\beta_D \mathbf{T}_C = \int da\ \beta_a \mathbf{T}_a, \tag{63}$$

with

$$\mathbf{T}_a(\mathbf{x},a,t) = -\frac{a}{\beta_a} \int_{|\mathbf{y}-\mathbf{x}|=a} dS_y\ \mathbf{nn} \int d^4p\ P(1;t) \times [\langle p_C \rangle_1(\mathbf{y},t|1) - \langle p_C \rangle(\mathbf{y},t)]. \tag{64}$$

It will be seen below that $\mathbf{T}_a$ has the physical meaning of a stress tensor in the liquid whose origin resides in the radial motion of the bubbles and their slip with respect to the liquid.

A similar procedure may be applied to approximate the term Q_a defined in Eq. (55) with the result

$$\beta_a Q_a(\mathbf{x},a,t) = \frac{a}{3} \int_{|\mathbf{y}-\mathbf{x}|=a} dS_y \int d^4p\ P(1;t)[\langle p_C \rangle_1(\mathbf{y},t|1) - \langle p_C \rangle(\mathbf{y},t)] + O(a^2/L^2). \tag{65}$$

Upon comparing with Eq. (64) one finds the interesting relation

$$Q_a(\mathbf{x},a,t) = -\tfrac{1}{3}\mathrm{Tr}[\mathbf{T}_a(\mathbf{x},a,t)], \tag{66}$$

which shows, from Eq. (56), that the pressure field surrounding the bubble centered at $\mathbf{x}$ differs from the unconditionally averaged pressure in the liquid by the contribution of the isotropic part of $\mathbf{T}_a$. This conforms precisely with the interpretation of this quantity as a stress tensor in the continuous phase expressing, at the level of the average quantities, the effects of the microscopic motion that have been lost due to averaging.

With these approximations the averaged equations of the previous section may be written in a slightly more familiar form as follows. The continuity equation for the continuous phase, Eq. (37), is unchanged and will not be repeated. The momentum equation (38) becomes instead

$$\rho_C \frac{\partial}{\partial t}(\beta_C\langle \mathbf{u}_C\rangle)+\rho_C\nabla\cdot(\beta_C\langle\mathbf{u}_C\rangle\langle\mathbf{u}_C\rangle)$$
$$=-\beta_C\nabla\langle p_C\rangle-\beta_D\mathbf{A}_D(\mathbf{x},t)+\nabla\cdot(\beta_C\rho_C\mathbf{M}_C+\beta_D\mathbf{T}_C)$$
$$+\beta_C\rho_C\mathbf{g}. \tag{67}$$

By taking $g=v_a$ in Eq. (31) and using the approximation (58) we find

$$\frac{\partial\beta_a}{\partial t}+\nabla\cdot(\beta_a\bar{\mathbf{w}}^a)+\frac{\partial}{\partial a}(\beta_a\bar{\dot{a}}^a)=4\pi a^2\bar{\dot{a}}^a n_a, \tag{68}$$

or, with Eq. (59),

$$\frac{\partial\beta_a}{\partial t}+\nabla\cdot(\beta_a\langle\mathbf{u}_a\rangle)+\frac{\partial}{\partial a}(\beta_a\bar{\dot{a}}^a)=4\pi a^2\bar{\dot{a}}^a n_a. \tag{69}$$

Upon integrating over all radii, by Eqs. (10) and (60), we have

$$\frac{\partial\beta_D}{\partial t}+\nabla\cdot(\beta_D\langle\mathbf{u}_D\rangle)=\int da\; n_a\dot{v}_a. \tag{70}$$

The small-particle approximation does not affect the gas momentum equation (50) while, by Eq. (66), the normal stress balance (54) becomes

$$\bar{p}_B^a(\mathbf{x},a,t)=\frac{2\sigma}{a}+\langle p_C\rangle(\mathbf{x},t)-\frac{1}{3}\mathrm{Tr}[\mathbf{T}_a(\mathbf{x},a,t)]. \tag{71}$$

The error estimates given before are only applicable to the case of strictly spherical bubbles. In the presence of deformations the errors acquire contributions of the order of a/L times a dimensionless measure of the bubble's deformation.

V. AVERAGE EQUATIONS TO FIRST ORDER OF β_D

We now show how the previous set of equations can be closed in the dilute limit. This is certainly the simplest case in which closure can be explicitly obtained. In addition, this analysis will help interpret the structure of the averaged equations in physical terms.

Since in the liquid momentum equation $\langle p_C\rangle_1$ appears in a term of $O(\beta_D)$ [cf. the right-hand side of Eq. (41)], for results accurate to $O(\beta_D)$, it is sufficient to calculate this quantity correct to $O(1)$. The pertinent equations are the same as given in Ref. 1, i.e.,

$$\nabla\cdot\langle\mathbf{u}_C\rangle_1=0, \tag{72}$$

$$\rho_C\left[\frac{\partial\langle\mathbf{u}_C\rangle_1}{\partial t}+\langle\mathbf{u}_C\rangle_1\cdot\nabla\langle\mathbf{u}_C\rangle_1\right]+\nabla\langle p_C\rangle_1=0. \tag{73}$$

All the fields are averaged conditionally to the presence of a bubble centered at $\mathbf{y}$ with radius a and generalized momentum $\mathbf{p}$. For $|\mathbf{x}-\mathbf{y}|\to\infty$, the effect of the bubble at $\mathbf{y}$ gets weaker and weaker so that[4]

$$\langle\mathbf{u}_C\rangle_1\to\langle\mathbf{u}_C\rangle,\quad \langle p_C\rangle_1\to\langle p_C\rangle. \tag{74}$$

Upon taking the conditional average of Eq. (24), we find the following form of the kinematic boundary condition to be applied at the surface of the bubble

$$\langle\mathbf{u}_C\rangle_1\cdot\mathbf{n}=\mathbf{w}\cdot\mathbf{n}+\dot{a}. \tag{75}$$

As shown in Ref. 1, to $O(\beta_D)$, both $\langle\mathbf{u}_C\rangle$ and $\langle\mathbf{u}_C\rangle_1$ are irrotational so that

$$\langle\mathbf{u}_C\rangle=\nabla\phi,\quad \langle\mathbf{u}_C\rangle_1=\nabla\phi_1. \tag{76}$$

The solution of the problem for ϕ_1 to the order required is given in the Appendix. The basic idea is to effect a Taylor series expansion of ϕ around the center of the bubble at $\mathbf{y}$. With this result all the quantities needed to close the problem can be calculated and will now be given in the small-particle approximation. It is clear from the order of the error terms indicated in the pertinent equations of Sec. IV that the terms neglected in this approximation are consistent with the present $O(\beta_D)$ accuracy.

The integrations necessary for the calculation of the the small-particle approximation expression (61) of $\mathbf{A}_a$ can be carried out with the result

$$\beta_a\mathbf{A}_a=\frac{1}{2}\rho_C\beta_a\left\{\frac{\partial\langle\mathbf{u}_C\rangle}{\partial t}+\langle\mathbf{u}_C\rangle\cdot\nabla\langle\mathbf{u}_C\rangle-\frac{\partial\bar{\mathbf{w}}^a}{\partial t}-\bar{\mathbf{w}}^a\cdot\nabla\bar{\mathbf{w}}^a\right\}$$
$$+\frac{1}{2}\rho_C n_a\bar{\dot{a}}^a\frac{\partial}{\partial a}[v_a(\langle\mathbf{u}_C\rangle-\bar{\mathbf{w}}^a)]+\frac{1}{2}\rho_C\nabla\cdot(\beta_a\mathbf{M}_a)$$
$$+\frac{1}{2}\rho_C\frac{\partial}{\partial a}(\beta_a\mathbf{N}_a). \tag{77}$$

Here, $\mathbf{M}_a$ is a Reynolds-like stress defined by

$$\mathbf{M}_a=\bar{\mathbf{w}}^a\bar{\mathbf{w}}^a-\overline{\mathbf{w}\mathbf{w}}^a=-\overline{(\mathbf{w}-\bar{\mathbf{w}}^a)(\mathbf{w}-\bar{\mathbf{w}}^a)}^a, \tag{78}$$

and $\mathbf{N}_a$ is given by

$$\mathbf{N}_a=\bar{\mathbf{w}}^a\bar{\dot{a}}^a-\overline{\mathbf{w}\dot{a}}^a=-\overline{(\dot{a}-\bar{\dot{a}}^a)(\mathbf{w}-\bar{\mathbf{w}}^a)}^a. \tag{79}$$

The last two terms in Eq. (77) represent the effect of the fluctuations in the translational and radial velocity of the particles on the added mass force. With the neglect of these terms, the result (77) conforms with the added mass force on a single sphere, the third term representing the effect of the variable radius. This point will be seen more clearly in Sec. VI.

Once $\langle p_C\rangle_1-\langle p_C\rangle$ has been determined, $\mathbf{T}_a$ can also be calculated from Eq. (64). The result is

$$\frac{1}{\rho_C}\mathbf{T}_a=\left[\frac{2}{5}(\bar{\mathbf{w}}^a-\langle\mathbf{u}_C\rangle)\cdot(\bar{\mathbf{w}}^a-\langle\mathbf{u}_C\rangle)-a\bar{\ddot{a}}^a-\frac{3}{2}(\bar{\dot{a}}^a)^2\right.$$
$$\left.+\frac{3}{2}D_a-\frac{2}{5}\mathrm{Tr}(\mathbf{M}_a)\right]\mathbf{I}-\frac{9}{20}(\bar{\mathbf{w}}^a-\langle\mathbf{u}_C\rangle)$$
$$\times(\bar{\mathbf{w}}^a-\langle\mathbf{u}_C\rangle)+\frac{9}{20}\mathbf{M}_a, \tag{80}$$

where $\mathbf{I}$ denotes the identity tensor and

$$D_a=(\bar{\dot{a}}^{a})^2-\overline{\dot{a}^2}^{a}=-\overline{(\dot{a}-\bar{\dot{a}}^{a})^2}^{a}. \tag{81}$$

The continuous-phase Reynolds stress is given by

$$\mathbf{M}_C=-\int da\ \beta_a\Bigg\{\Bigg[\Bigg[(\bar{\dot{a}}^{a})^2+\frac{3}{20}\left(\langle\mathbf{u}_C\rangle-\bar{\mathbf{w}}^a\right)\cdot\left(\langle\mathbf{u}_C\rangle-\bar{\mathbf{w}}^a\right)\Bigg]\mathbf{I}+\frac{1}{20}\left(\langle\mathbf{u}_C\rangle-\bar{\mathbf{w}}^a\right)\left(\langle\mathbf{u}_C\rangle-\bar{\mathbf{w}}^a\right)-\Bigg[\left(\frac{3}{20}\left(\mathrm{Tr}\ \mathbf{M}_a\right)+D_a\right)\mathbf{I}+\frac{1}{20}\mathbf{M}_a\Bigg]\Bigg]\Bigg\}. \tag{82}$$

The result for the quantity Q_a defined by Eq. (55) is readily obtained by taking the trace of $\mathbf{T}_a$ according to Eq. (66) and is

$$Q_a=\rho_C[a\bar{\ddot{a}}^{a}+\tfrac{3}{2}(\bar{\dot{a}}^{a})^2-\tfrac{1}{4}(\bar{\mathbf{w}}^a-\langle\mathbf{u}_C\rangle)(\bar{\mathbf{w}}^a-\langle\mathbf{u}_C\rangle)-\tfrac{3}{2}D_a+\tfrac{1}{4}\mathrm{Tr}(\mathbf{M}_a)]. \tag{83}$$

These expressions have been obtained on the assumption of potential flow which evidently fails were the system to be observed from a rotating coordinate frame. However, as shown in Ref. 1, they are easily corrected so as to be applicable to such a situation by the addition to $\mathbf{A}_a$ of the lift force

$$\mathbf{L}_a=\rho_C(\bar{\mathbf{w}}^a-\langle\mathbf{u}_C\rangle)\times\mathbf{\Omega}, \tag{84}$$

where $\mathbf{\Omega}$ is the angular velocity of the rotating frame. It was shown in Ref. 1 that this expression also holds if the mixture rotates in solid body motion. The quantity $\mathbf{A}_C$ is similarly modified to

$$\mathbf{A}'_C=\mathbf{A}_C-\frac{1}{\beta}\int da\ \beta_a\mathbf{L}_a. \tag{85}$$

If the angular velocity is not truly uniform in space, but slowly varying as would happen in the case of a flow with regions of nonzero vorticity large compared with a bubble diameter, it appears reasonable that the same corrections should be introduced with $\mathbf{\Omega}$ replaced by $-\tfrac{1}{2}\nabla\times\langle\mathbf{u}_C\rangle$. Further considerations on this point can be found in Ref. 1.

In this way, combining all the previous results, we have the final form of the momentum equations correct to $O(\beta_D)$. For the continuous phase the result is

$$\begin{aligned}\rho_C\beta_C\left[\frac{\partial\langle\mathbf{u}_C\rangle}{\partial t}+(\langle\mathbf{u}_C\rangle\cdot\nabla)\langle\mathbf{u}_C\rangle\right]+\beta_C\nabla\langle p_C\rangle=&\frac{1}{2}\rho_C\int da\ n_a\Bigg\{v_a\left[\frac{\partial\bar{\mathbf{w}}^a}{\partial t}+\bar{\mathbf{w}}^a\cdot\nabla\bar{\mathbf{w}}^a-\frac{\partial\langle\mathbf{u}_C\rangle}{\partial t}-\langle\mathbf{u}_C\rangle\cdot\nabla\langle\mathbf{u}_C\rangle\right]\\&+\bar{\dot{a}}^a\frac{\partial}{\partial a}[v_a(\bar{\mathbf{w}}^a-\langle\mathbf{u}_C\rangle)]-v_a(\nabla\times\langle\mathbf{u}_C\rangle)\times(\bar{\mathbf{w}}^a-\langle\mathbf{u}_C\rangle)\Bigg\}\\&+\rho_C\nabla\cdot\int da\ \beta_a\{[\tfrac{1}{4}(\bar{\mathbf{w}}^a-\langle\mathbf{u}_C\rangle)\cdot(\bar{\mathbf{w}}^a-\langle\mathbf{u}_C\rangle)-a\bar{\ddot{a}}^{a}-\tfrac{5}{2}\overline{\dot{a}^2}^{a}]\mathbf{I}\\&-\tfrac{1}{2}(\bar{\mathbf{w}}^a-\langle\mathbf{u}_C\rangle)(\bar{\mathbf{w}}^a-\langle\mathbf{u}_C\rangle)-\tfrac{1}{4}\mathrm{Tr}(\mathbf{M}_a)\mathbf{I}\}.\end{aligned} \tag{86}$$

The momentum equation (50) for the fraction of the gas phase in the form of bubbles of radius a can be written as

$$\begin{aligned}\beta_a\nabla\langle p_C\rangle=&\frac{1}{2}\rho_C\beta_a\left\{\frac{\partial\langle\mathbf{u}_C\rangle}{\partial t}+\langle\mathbf{u}_C\rangle\cdot\nabla\langle\mathbf{u}_C\rangle-\frac{\partial\bar{\mathbf{w}}^a}{\partial t}-\bar{\mathbf{w}}^a\cdot\nabla\bar{\mathbf{w}}^a\right\}+\frac{1}{2}\rho_C\beta_a(\nabla\times\langle\mathbf{u}_C\rangle)\times(\bar{\mathbf{w}}^a-\langle\mathbf{u}_C\rangle)+\frac{1}{2}\rho_Cn_a\bar{\dot{a}}^a\\&\times\frac{\partial}{\partial a}[v_a(\langle\mathbf{u}_C\rangle-\bar{\mathbf{w}}^a)]+\frac{1}{2}\rho_C\left[\nabla\cdot(\beta_a\mathbf{M}_a)+\frac{\partial}{\partial a}(\beta_a\mathbf{N}_a)\right].\end{aligned} \tag{87}$$

After using the expression (83) for Q_a and the relation (33) to express $\bar{\ddot{a}}$, Eq. (56) for the radial motion can be written as

$$\begin{aligned}&a\left[\frac{\partial\bar{\dot{a}}^a}{\partial t}+\bar{\mathbf{w}}^a\cdot\nabla\bar{\dot{a}}^a+\bar{\dot{a}}^a\frac{\partial\bar{\dot{a}}^a}{\partial a}\right]+\frac{3}{2}(\bar{\dot{a}}^a)^2\\&=\frac{1}{\rho_C}\left[\overline{p_B}^a-\frac{2\sigma}{a}-\langle p_C\rangle\right]+\frac{1}{4}(\bar{\mathbf{w}}^a-\langle\mathbf{u}_C\rangle)\cdot(\bar{\mathbf{w}}^a-\langle\mathbf{u}_C\rangle)+\frac{3}{2}D_a-\frac{1}{4}\mathrm{Tr}\ \mathbf{M}_a.\end{aligned} \tag{88}$$

This equation exhibits of course a strong similarity with the well-known Rayleigh–Plesset equation for the radial dynamics of single bubbles. The radial acceleration is expressed in terms of the convective derivative of $\dot{a}$ in phase space. In addition, the average ambient pressure in the liquid is modified by the relative motion between the bubbles and the host liquid—the term multiplied by the factor of 1/4 is indeed the average surface pressure over a translating sphere—and by the microscopic "disordered" motion of the other bubbles as expressed by the last two terms. All of these effects are due to the additional stress tensor $\mathbf{T}_a$ which, once more, is seen to be endowed with a very concrete physical reality.

In principle one pair of Eqs. (87) and (88) must be solved for each value of a in the mixture. Such level of detail is probably both impractical and excessive for most applications. It is therefore interesting to exhibit explicitly the corresponding equations after integration over the radius distribution. These cannot be derived simply by integration of Eqs. (87) and (88) because of the presence of the nonlinear terms. It is necessary to write the derivatives of $\bar{\mathbf{w}}^a$ in the first line in conservation form, use the definition of $\mathbf{M}_a$ to return to the average of products in place of the product of averages, and to use Eq. (68) for β_a. For Eq. (87) the final result is

$$\beta_D \nabla \langle p_C \rangle = \frac{1}{2} \rho_C \beta_D \left[\frac{\partial \langle \mathbf{u}_C \rangle}{\partial t} + \langle \mathbf{u}_C \rangle \cdot \nabla \langle \mathbf{u}_C \rangle - \frac{\partial \langle \mathbf{u}_D \rangle}{\partial t} - \langle \mathbf{u}_D \rangle \cdot \nabla \langle \mathbf{u}_D \rangle \right] + \frac{1}{2} \rho_C \beta_D (\nabla \times \langle \mathbf{u}_C \rangle) \times (\langle \mathbf{u}_D \rangle - \langle \mathbf{u}_C \rangle)$$
$$- \frac{1}{2} \rho_C \int (\bar{\mathbf{w}}^a - \langle \mathbf{u}_C \rangle) da \; n_a \overline{\dot{v}_a}^{a} + \frac{1}{2} \rho_C \nabla \cdot (\beta_D \mathbf{M}_D). \tag{89}$$

Here, $\langle \mathbf{u}_D \rangle$ is defined by Eq. (60) after the small-particle approximation (59), and

$$\mathbf{M}_D = \langle \mathbf{u}_D \rangle \langle \mathbf{u}_D \rangle - \langle \mathbf{u}_D \mathbf{u}_D \rangle = -\langle (\mathbf{u}_D - \langle \mathbf{u}_D \rangle)(\mathbf{u}_D - \langle \mathbf{u}_D \rangle) \rangle. \tag{90}$$

It will be clearly seen in the next section that the remaining integral in Eq. (89) accounts for the effect of the volume variation of the bubbles on the added mass force.

The equation of motion (88) for the bubble radius does not simplify appreciably upon integration over a. The result is

$$\overline{p_B} = \frac{2\sigma}{n} \int \frac{n_a}{a} da + \langle p_C \rangle + \rho_C [\overline{a\ddot{a}} + \tfrac{3}{2}\overline{\dot{a}^2} - \tfrac{1}{4}(\langle \mathbf{u}_D \rangle - \langle \mathbf{u}_C \rangle) \cdot (\langle \mathbf{u}_D \rangle - \langle \mathbf{u}_C \rangle) + \tfrac{1}{4} \mathrm{Tr}\, \mathbf{M}_D]. \tag{91}$$

With procedures similar to those used to obtain Eq. (89), the continuous phase momentum equation (86) can be written in a somewhat more compact form as

$$\rho_C \beta_C \left[\frac{\partial \langle \mathbf{u}_C \rangle}{\partial t} + (\langle \mathbf{u}_C \rangle \cdot \nabla) \langle \mathbf{u}_C \rangle \right] + \beta_C \nabla \langle p_C \rangle$$
$$= -\frac{1}{2} \rho_C \beta_D \left[\frac{\partial \langle \mathbf{u}_C \rangle}{\partial t} + \langle \mathbf{u}_C \rangle \cdot \nabla \langle \mathbf{u}_C \rangle - \frac{\partial \langle \mathbf{u}_D \rangle}{\partial t} - \langle \mathbf{u}_D \rangle \cdot \nabla \langle \mathbf{u}_D \rangle \right] + \frac{1}{2} \rho_C \int (\bar{\mathbf{w}}^a - \langle \mathbf{u}_C \rangle) da \; n_a \overline{\dot{v}_a}^{a} - \frac{1}{2} \rho_C \beta_D (\nabla \times \langle \mathbf{u}_C \rangle)$$
$$\times (\langle \mathbf{u}_D \rangle - \langle \mathbf{u}_C \rangle) + \rho_C \nabla \cdot \left\{ \frac{1}{4} \beta_D [(\langle \mathbf{u}_D \rangle - \langle \mathbf{u}_C \rangle) \cdot (\langle \mathbf{u}_D \rangle - \langle \mathbf{u}_C \rangle) - \mathrm{Tr}\, \mathbf{M}_D] \mathbf{I} - \mathbf{I} \int da \; \beta_a \left[\frac{5}{2} \overline{\dot{a}^2}^{a} + a \overline{\ddot{a}}^{a} \right] - \frac{1}{2} \beta_D (\langle \mathbf{u}_D \rangle \right.$$
$$\left. - \langle \mathbf{u}_C \rangle)(\langle \mathbf{u}_D \rangle - \langle \mathbf{u}_C \rangle) + \frac{1}{2} \left[\int da \; \beta_a \mathbf{M}_a - \beta_D \mathbf{M}_D \right] \right\}. \tag{92}$$

A comparison with definition (78) shows that, in general, the last two terms in the square brackets do not cancel.

We derive in the next section the limiting form of these integrated equations in the case of a narrowly peaked probability distribution.

VI. NARROWLY PEAKED PROBABILITY DISTRIBUTION

Although in the present paper we only consider bubbles with an identical equilibrium radius, for each realization with a bubble at some position $\mathbf{y}$ we must allow for different radii and radial velocities because the state of the mixture in the neighborhood of $\mathbf{y}$ can be different in different realizations. For this reason many of the formulas given in the previous section still involve integrals over a and $\dot{a}$. In order to simplify these relations and to gain a better insight into their physical content it is interesting to consider the limit in which the probability distribution of a and $\dot{a}$ is narrowly peaked around the average values $\bar{a}(\mathbf{y},t)$ and $\bar{\dot{a}}(\mathbf{y},t)$. To study this case we assume that

$$P(1;t) \equiv P(\mathbf{y},a,\mathbf{w},\dot{a};t) \propto \delta[a - \bar{a}(\mathbf{y},t)] \delta[\dot{a} - \bar{\dot{a}}(\mathbf{y},t)]. \tag{93}$$

With this assumption the various integrations over a and $\dot{a}$ that appear in the previous formulas are readily carried out. For example, Eq. (11) shows that

$$n_a(\mathbf{y},a,t) = n(\mathbf{y},t) \delta[a - \bar{a}(\mathbf{y},t)]. \tag{94}$$

With Eq. (93) it is also easy to deduce from Eq. (30) that $\overline{a\mathbf{w}} = \bar{a}\bar{\mathbf{w}}$ so that Eq. (45) gives

$$\bar{\dot{a}} = \frac{\partial \bar{a}}{\partial t} + \bar{\mathbf{w}} \cdot \nabla \bar{a}, \tag{95}$$

and, similarly,

$$\bar{\ddot{a}} = \frac{\partial \bar{\dot{a}}}{\partial t} + \bar{\mathbf{w}} \cdot \nabla \bar{\dot{a}}. \tag{96}$$

For simplicity, in the following we drop the averaging indication writing, e.g., $a(\mathbf{x},t)$ in place of $\bar{a}(\mathbf{x},t)$, $\mathbf{u}_C$ in place of $\langle \mathbf{u}_C \rangle$, etc.

The continuous-phase continuity equation (37) remains unchanged. The disperse phase one, Eq. (70), gives

$$\frac{\partial \beta_D}{\partial t} + \nabla \cdot (\beta_D \mathbf{w}) = 4\pi n a^2 \dot{a}. \tag{97}$$

The liquid momentum equation (92) becomes

$$\rho_C\beta_C\left[\frac{\partial \mathbf{u}_C}{\partial t}+(\mathbf{u}_C\cdot\nabla)\mathbf{u}_C\right]+\beta_C\nabla p_C$$
$$=-\frac{1}{2}\rho_C\beta_D\left[\frac{\partial \mathbf{u}_C}{\partial t}+\mathbf{u}_C\cdot\nabla\mathbf{u}_C-\frac{\partial \mathbf{w}}{\partial t}-\mathbf{w}\cdot\nabla\mathbf{w}\right]$$
$$+\frac{1}{2}\rho_C n\dot{v}_a(\mathbf{w}-\mathbf{u}_C)-\frac{1}{2}\rho_C\beta_D(\nabla\times\mathbf{u}_C)\times(\mathbf{w}-\mathbf{u}_C)$$
$$+\rho_C\nabla\cdot\left\{\frac{1}{4}\beta_D[(\mathbf{w}-\mathbf{u}_C)\cdot(\mathbf{w}-\mathbf{u}_C)-\mathrm{Tr}\,\mathbf{M}_D]\mathbf{I}-\beta_D\right.$$
$$\times\left[\frac{5}{2}\dot{a}^2+a\left(\frac{\partial \dot{a}}{\partial t}+\mathbf{w}\cdot\nabla\dot{a}\right)\right]\mathbf{I}-\frac{\beta_D}{2}(\mathbf{w}-\mathbf{u}_C)$$
$$\left.\times(\mathbf{w}-\mathbf{u}_C)\right\}. \tag{98}$$

The disperse phase momentum equation (89) becomes

$$\beta_D\nabla p_C=\frac{1}{2}\rho_C\beta_D\left[\frac{\partial \mathbf{u}_C}{\partial t}+\mathbf{u}_C\cdot\nabla\mathbf{u}_C-\frac{\partial \mathbf{w}}{\partial t}-\mathbf{w}\cdot\nabla\mathbf{w}\right]$$
$$-\frac{1}{2}\rho_C n\dot{v}_a(\mathbf{w}-\mathbf{u}_C)+\frac{1}{2}\rho_C\beta_D(\nabla\times\mathbf{u}_C)$$
$$\times(\mathbf{w}-\mathbf{u}_C)+\frac{1}{2}\rho_C\nabla\cdot(\beta_D\mathbf{M}_D). \tag{99}$$

With these results one can get a better understanding of the effect of bubble volume variation on the added mass interaction. The relevant terms, appearing with opposite sign in Eqs. (98) and (99), are, using $\beta_D\simeq nv_a$

$$\frac{1}{2}nv_a\left[\frac{\partial \mathbf{w}}{\partial t}+\mathbf{w}\cdot\nabla\mathbf{w}-\frac{\partial \mathbf{u}_C}{\partial t}-\mathbf{u}_C\cdot\nabla\mathbf{u}_C\right]+\frac{1}{2}n\dot{v}_a(\mathbf{w}-\mathbf{u}_C). \tag{100}$$

For a single bubble one would write $(d/dt)[\frac{1}{2}v_a(\mathbf{w}-\mathbf{u}_C)]$. It is evident that (100) is in precise correspondence with this relation. The total time derivatives are interpreted as convective derivatives with the continuous phase velocity for $\mathbf{u}_C$, and with the bubble velocity for v_a and $\mathbf{w}$.

The bubble radial equation (91) is

$$a\left(\frac{\partial \dot{a}}{\partial t}+\mathbf{w}\cdot\nabla\dot{a}\right)+\frac{3}{2}\dot{a}^2$$
$$=\frac{1}{\rho_C}\left[p_B-\frac{2\sigma}{a}-p_C+\frac{1}{4}\rho_C(\mathbf{w}-\mathbf{u}_C)(\mathbf{w}-\mathbf{u}_C)\right.$$
$$\left.-\frac{1}{4}\rho_C\,\mathrm{Tr}\,\mathbf{M}_D\right], \tag{101}$$

with $\dot{a}$ given by Eq. (95).

VII. DESCRIPTION IN TERMS OF THE IMPULSE

A description in terms of the impulse is sometimes favored for dispersions of particles or bubbles in potential flow.[10,11] We have recently presented an analysis of this description showing that, for rigid spheres, it is exactly equivalent to the one obtained by our methods.[2] Here, we add some further comments for the present case of massless spheres with a variable radius.

Before starting, and only for the purposes of this section, we need to introduce some further notation. Define the contribution of the bubbles of radius a to the liquid volume fraction by

$$\beta_C^a(\mathbf{x},t|a)=\frac{1}{N!}\int d^3y\int d^4p\int d\mathscr{C}^{N-1}\,\chi_C(\mathbf{x};N)P(N;t), \tag{102}$$

and, similarly, their contribution to the continuous-phase average of any quantity f_C by

$$\langle f_C\rangle^a(\mathbf{x},t|a)=\frac{1}{\beta_C^a N!}\int d^3y\int d^4p\int d\mathscr{C}^{N-1}$$
$$\times f_C(\mathbf{x},t;N)\chi_C(\mathbf{x};N)P(N;t). \tag{103}$$

By using the Bernoulli integral the bubble equation of motion (47) becomes

$$0=-\rho_C\int_{|\mathbf{x}-\mathbf{z}|=a}\left(\frac{d\phi_C}{dt}-\frac{1}{2}u_C^2\right)\mathbf{n}\,dS_z, \tag{104}$$

where $d\phi_C/dt$ is the convective derivative

$$\frac{d\phi_C}{dt}=\frac{\partial\phi_C}{\partial t}+\mathbf{u}_C\cdot\nabla\phi_C=\frac{\partial\phi_C}{\partial t}+u_C^2. \tag{105}$$

Denote now by $\mathbf{J}$ the impulse, defined by

$$\mathbf{J}=-\rho_C\int_{|\mathbf{x}-\mathbf{z}|=a}dS_z\,\mathbf{n}\phi_C. \tag{106}$$

By applying a standard transport theorem (see, e.g., Refs. 12 and 13) the total derivative of $\mathbf{J}$ can be calculated as

$$\frac{d\mathbf{J}}{dt}=-\int_{|\mathbf{z}-\mathbf{x}|=a}\frac{d\phi_C}{dt}\mathbf{n}\,dS_z$$
$$+\int_{|\mathbf{z}-\mathbf{x}|=a}\phi_C(\nabla\mathbf{u}_C)\cdot\mathbf{n}\,dS_z. \tag{107}$$

The last term can be rewritten as

$$\int_{|\mathbf{z}-\mathbf{x}|=a}[\partial_i(\phi_C\partial_j\phi_C)n_j-\partial_i\phi_C\partial_j\phi_C n_j]dS_z$$
$$=\int_{|\mathbf{z}-\mathbf{x}|=a}(u_C^2\delta_{ij}-u_{Ci}u_{Cj})n_j\,dS_z, \tag{108}$$

where we have used the following identity which holds for any closed surface:

$$\int\partial_i(\phi_C\partial_j\phi_C)n_j\,dS_z=\int\partial_j(\phi_C\partial_j\phi_C)n_i\,dS_z. \tag{109}$$

With this, combining with Eq. (104), we obtain

$$\frac{d\mathbf{J}}{dt}=\rho_C\int_{|\mathbf{z}-\mathbf{x}|=a}dS_z\left(\frac{1}{2}u_C^2\mathbf{I}-\mathbf{u}_C\mathbf{u}_C\right)\cdot\mathbf{n}. \tag{110}$$

This relation is now averaged according to Eq. (28) to find

$$n_a \frac{\overline{d\mathbf{J}}^{a}}{dt} = \rho_C \int_{|\mathbf{z}-\mathbf{x}|=a} dS_z \int d^4p\ P(\mathbf{x},a,\mathbf{p};t)\mathbf{n} \cdot \langle \tfrac{1}{2} u_C^2 \mathbf{I} - \mathbf{u}_C \mathbf{u}_C \rangle_1, \tag{111}$$

where we have used the fact that $\beta_C^1 = 1$ on the particle surface.

For potential flow $\nabla \cdot (\frac{1}{2} u_C^2 \mathbf{I} - \mathbf{u}_C \mathbf{u}_C) = 0$. It is readily shown following the same steps leading to Eq. (17) that

$$\nabla (\beta_C^a \langle f_C \rangle^a) = \beta_C^a \langle \nabla f_C \rangle^a + \int_{|\mathbf{y}-\mathbf{x}|=a} dS_z \int d^4p\ dS_y \times P(\mathbf{y},a,\mathbf{p};t)\mathbf{n} \langle f_C \rangle_1 (\mathbf{x},t|\mathbf{y},a,\mathbf{p}), \tag{112}$$

so that

$$\nabla \left(\beta_C^a \left\langle \frac{1}{2} u_C^2 \mathbf{I} - \mathbf{u}_C \mathbf{u}_C \right\rangle^a \right) - \int_{|\mathbf{y}-\mathbf{x}|=a} dS_y \int d^4p\ P(\mathbf{y},a,\mathbf{p};t)\mathbf{n} \cdot \langle \tfrac{1}{2} u_C^2 \mathbf{I} - \mathbf{u}_C \mathbf{u}_C \rangle_1 = 0. \tag{113}$$

Upon adding this identity to Eq. (111) one finds

$$n_a \frac{\overline{d\mathbf{J}}^{a}}{dt} = \rho_C \nabla \left(\beta_C^a \left\langle \frac{1}{2} u_C^2 \mathbf{I} - \mathbf{u}_C \mathbf{u}_C \right\rangle^a \right) + \rho_C \int_{|\mathbf{z}-\mathbf{x}|=a} dS_z \int d^4p\ P(\mathbf{x},a,\mathbf{p};t)\mathbf{n} \cdot \left\langle \frac{1}{2} u_C^2 \mathbf{I} - \mathbf{u}_C \mathbf{u}_C \right\rangle_1 - \rho_C \int_{|\mathbf{y}-\mathbf{x}|=a} dS_y \int d^4p \times P(\mathbf{y},a,\mathbf{p};t)\mathbf{n} \cdot \left\langle \frac{1}{2} u_C^2 \mathbf{I} - \mathbf{u}_C \mathbf{u}_C \right\rangle_1. \tag{114}$$

The two integrals in the right-hand side do not exactly cancel. However, by using the same procedure leading to Eq. (62) and the transport theorem (34), we obtain

$$n_a \left[\frac{\partial \bar{\mathbf{J}}^a}{dt} + \bar{\mathbf{w}}^a \cdot \nabla \bar{\mathbf{J}}^a \right] = -\nabla \cdot (n_a \tau_k^a + \sigma^a), \tag{115}$$

where

$$n_a \tau_k^a = n_a (\overline{\mathbf{wJ}}^{a} - \bar{\mathbf{w}}^a \bar{\mathbf{J}}^a), \tag{116}$$

is the kinetic contribution to the particle stress and

$$\sigma^a = \rho_C \beta_C^a \langle \mathbf{u}_C \mathbf{u}_C \rangle^a - \frac{1}{2} \rho_C \beta_C^a \langle u_C^2 \rangle^a \mathbf{I} - \rho_C a \int d^4p\ P(\mathbf{x},a,\mathbf{p};t) \int_{|\mathbf{y}-\mathbf{x}|=a} dS_z \int d^4p\ \mathbf{nn} \times \left\langle \frac{1}{2} u_C^2 \mathbf{I} - \mathbf{u}_C \mathbf{u}_C \right\rangle_1 (\mathbf{y},t|\mathbf{x},a,\mathbf{p}), \tag{117}$$

may be identified with the potential contribution. By following the same procedure outlined in Appendix C of Ref. 1, the same result can also be shown to follow from the equations of Sec. IV.

After integration over a we find

$$n \left[\frac{\partial \bar{\mathbf{J}}}{dt} + \bar{\mathbf{w}} \cdot \nabla \bar{\mathbf{J}} \right] = -\nabla \cdot (n \tau_k + \sigma), \tag{118}$$

where

$$n \tau_k = -n(\overline{\mathbf{wJ}} - \bar{\mathbf{w}} \bar{\mathbf{J}}), \tag{119}$$

$$\sigma = \rho_C \beta_C \langle \mathbf{u}_C \mathbf{u}_C \rangle - \frac{1}{2} \rho_C \beta_C \langle u_C^2 \rangle \mathbf{I} - \rho_C \int da\ a \int d^4p\ P(\mathbf{x},a,\mathbf{p};t) \int_{|\mathbf{y}-\mathbf{x}|=a} dS_z \int d^4p \mathbf{nn} \times \left\langle \frac{1}{2} u_C^2 \mathbf{I} - \mathbf{u}_C \mathbf{u}_C \right\rangle_1 (\mathbf{y},t|\mathbf{x},a,\mathbf{p}). \tag{120}$$

For a narrowly peaked probability distribution of the type of Eq. (93), this result is formally identical to the one found in the rigid-particle case.[2]

It may be of some interest to consider explicitly the dilute-limit case. By using the expression given in the Appendix for the potential $\langle \phi_C \rangle_1$ one readily finds

$$\bar{\mathbf{J}}^a = \rho_C v_a \left(\frac{1}{2} \bar{\mathbf{w}}^a - \frac{3}{2} \langle \mathbf{u}_C \rangle \right). \tag{121}$$

Furthermore, again with the results of the Appendix, one can prove that

$$\overline{\mathbf{wJ}}^{a} = \rho_C \left(\frac{1}{2} \bar{\mathbf{w}}^a \bar{\mathbf{w}}^a - \frac{3}{2} \bar{\mathbf{w}}^a \langle \mathbf{u}_C \rangle \right) - \frac{1}{2} \rho_C v_a \mathbf{M}_a. \tag{122}$$

Upon averaging over a these expressions become, respectively,

$$\bar{\mathbf{J}} = \rho_C v \left(\frac{1}{2} \bar{\mathbf{w}} - \frac{3}{2} \langle \mathbf{u}_C \rangle \right), \tag{123}$$

$$\overline{\mathbf{wJ}} = \rho_C \left(\frac{1}{2} \bar{\mathbf{w}} \bar{\mathbf{w}} - \frac{3}{2} \bar{\mathbf{w}} \langle \mathbf{u}_C \rangle \right) - \frac{1}{2} \rho_C \mathbf{M}_D. \tag{124}$$

To the required order, the integral in Eq. (120) may be evaluated by replacing $\mathbf{u}_C$ by $\langle \mathbf{u}_C \rangle_1$ and one finds

$$\frac{\sigma}{\rho_C} = \rho_C \beta_C \left(\langle \mathbf{u}_C \rangle \langle \mathbf{u}_C \rangle - \frac{1}{2} \langle \mathbf{u}_C \rangle \cdot \langle \mathbf{u}_C \rangle \mathbf{I} \right) + \beta_D \rho_C \left[\bar{\mathbf{w}} \langle \mathbf{u}_C \rangle + \frac{1}{2} \langle \mathbf{u}_C \rangle (\langle \mathbf{u}_C \rangle - \bar{\mathbf{w}}) + \langle \mathbf{u}_C \rangle \cdot \left(\frac{1}{2} \bar{\mathbf{w}} - \langle \mathbf{u}_C \rangle \right) \mathbf{I} \right]. \tag{125}$$

Upon substituting Eqs. (123), (124), and (125) into Eqs. (118) and (119) we find the same equation that follows upon elimination of $\nabla \langle p_C \rangle$ between Eqs. (98) and (99) without the lift terms.

VIII. LINEAR PRESSURE WAVES

As an application of the equations of the previous section we now consider the propagation of small-amplitude pressure waves in a bubbly liquid.

Upon linearization, the liquid momentum equation (98) is

$$\rho\beta_C \frac{\partial \mathbf{u}'}{\partial t} + \beta_C \nabla p' = -\frac{1}{2}\rho\beta_D\left(\frac{\partial \mathbf{u}'}{\partial t} - \frac{\partial \mathbf{w}'}{\partial t}\right) - \rho\nabla\left(\beta_D a_0 \frac{\partial^2 a'}{\partial t^2}\right), \tag{126}$$

where the prime denotes perturbations with respect to the undisturbed state (subscript 0). Also, for simplicity of writing, in this section we drop the subscript from ρ_C, p_C, and $\mathbf{u}_C$. Actually, the last term in the right-hand side of this equation, arising from the linearization of the stress tensor $\mathbf{T}_C$, should be discarded consistently with the small-particle approximation of Sec. IV. To prove this statement we estimate the order of magnitude of the liquid velocity from the equation of continuity (37) by setting $\nabla \cdot \mathbf{u}' \sim ku'$ and $\partial\beta_C/\partial t = -\partial\beta_D/\partial t \sim \omega\beta_D'$, where we envision a wavelike solution with wave number k and frequency ω. Ignoring for the moment the contribution to β_D' due to variations in the number density, we have $\beta_D' \sim \beta_D(a'/a_0)$, from which

$$u' \sim \beta_D \frac{\omega}{k}\frac{a'}{a_0}. \tag{127}$$

With this estimate, the ratio of the last to the first term in Eq. (126) is

$$\frac{\rho\nabla[\beta_D a_0(\partial^2 a'/\partial t^2)]}{\rho\beta_C(\partial \mathbf{u}'/\partial t)} \sim \frac{k\beta_D a_0 \omega^2 a'}{\omega u'} \sim k^2 a^2. \tag{128}$$

Since $1/k$ is one of the macroscopic length scales of the problem, this contribution is of the same order as those that have been discarded in Sec. IV. The same conclusion would apply in the presence of nonuniformities in the spatial distribution of β_D. Hence we drop this term in the following.

In order to make the later discussion more transparent it is useful to rewrite Eq. (126) in the form

$$\rho\beta_C \frac{\partial \mathbf{u}'}{\partial t} + \beta_C \nabla p' = -\frac{1}{2}\rho\beta_D C(\beta_D,\omega)\left(\frac{\partial \mathbf{u}'}{\partial t} - \frac{\partial \mathbf{w}'}{\partial t}\right) + ko(\beta_D). \tag{129}$$

Here, the coefficient C is a modified added mass coefficient that, in the incompressible limit (low frequencies) reduces to the one introduced in Ref. 1. For small β_D and ω it has the expansion[1]

$$C = 1 - 0.24\beta_D + O(\beta_D^2). \tag{130}$$

The error term $ko(\beta_D)$ is a consequence of the inaccuracy with which $\mathbf{T}_a$ has been determined in Sec. V with the factor k reflecting the presence of the divergence of this quantity in the momentum equation.

Similarly, we write the bubble momentum equation (99) in the form

$$\beta_D \nabla p' = \frac{1}{2}\rho\beta_D C_\rho\left(\frac{\partial \mathbf{u}'}{\partial t} - \frac{\partial \mathbf{w}'}{\partial t}\right), \tag{131}$$

with the same C and no error other than those introduced by linearization and the small-particle approximation.

By adding Eqs. (129) and (131) we find

$$\rho\beta_C \frac{\partial \mathbf{u}'}{\partial t} + \nabla p' = 0 + ko(\beta_D), \tag{132}$$

while, upon eliminating $\nabla p'$,

$$\frac{\partial \mathbf{w}'}{\partial t} = \frac{2\beta_C + C}{C}\frac{\partial \mathbf{u}'}{\partial t} + ko(\beta_D). \tag{133}$$

At low frequencies the influence of the error term becomes negligible and the incompressible approximation (130) to C can be used to find

$$\frac{\partial \mathbf{w}'}{\partial t} \simeq (3 - 1.52\beta_D)\frac{\partial \mathbf{u}'}{\partial t}. \tag{134}$$

By differentiating with respect to time the two continuity equations (37) and (97) we also find, to the linear approximation,

$$-\frac{\partial^2 \beta_D'}{\partial t^2} + \nabla \cdot \left(\beta_C \frac{\partial \mathbf{u}'}{\partial t}\right) = 0, \tag{135}$$

$$\frac{\partial^2 \beta_D'}{\partial t^2} + \nabla \cdot \left(\beta_D \frac{\partial \mathbf{w}'}{\partial t}\right) = 4\pi n_0 a_0^2 \frac{\partial^2 a'}{\partial t^2}. \tag{136}$$

Upon adding these relations and using Eqs. (132) and (133) to eliminate the velocity fields, one is led to

$$\nabla \cdot \left[\left(1 + \frac{\beta_D}{\beta_C}\frac{2\beta_C + C}{C}\right)\nabla p' + ko(\beta_D)\right] + \frac{3\beta_D \rho}{a_0}\frac{\partial^2 a'}{\partial t^2} = 0. \tag{137}$$

To close the problem we need to consider the radial equation (101). After linearizing and accounting for the equilibrium condition

$$p_{B0} = p_0 + \frac{2\sigma}{a_0}, \tag{138}$$

it becomes

$$p_B' = p' - \frac{2\sigma}{a_0^2}a' + \rho a_0 \frac{\partial^2 a'}{\partial t^2}[1 + o(1)], \tag{139}$$

where the error term reflects the order of accuracy of $\mathbf{T}_a$ and is therefore related to the one in Eq. (129).

To calculate p_B', following Refs. 6 and 14, we assume a time dependence proportional to $\exp i\omega t$ and write

$$p_B = p_{B0}\left(1 - \Phi \frac{a'}{a_0}\right), \tag{140}$$

where the complex function Φ is given by (see, e.g., Refs. 10 and 11)

$$\frac{3\gamma}{\Phi} = 1 - 3(\gamma - 1)i\chi\left[\sqrt{\frac{i}{\chi}}\coth\sqrt{\frac{i}{\chi}} - 1\right], \tag{141}$$

with $\chi = D/\omega a_0^2$, D the gas thermal diffusivity, and γ the ratio of specific heats. It may be shown that $\Phi \to 3$ and $\Phi \to 3\gamma$ at low and high frequencies, respectively. The bubble damping parameter b and natural frequency ω_0 are given by (Refs. 6 and 14)

$$b=\frac{p_{B0}}{2\rho a_0^2\omega}\,\mathrm{Im}\,\Phi, \quad \omega_0^2=\frac{p_{B0}}{\rho a_0^2}\left(\mathrm{Re}\,\Phi-\frac{2\sigma}{a_0 p_{B0}}\right), \tag{142}$$

where viscous and acoustic damping effects have been dropped in keeping with the approximations made in this paper. Upon substituting in Eq. (139) we then find

$$a'=-\frac{p'}{\rho a_0}\frac{1}{\omega_0^2+2ib\omega-\omega^2}\,[1+\omega^2 o(1)]. \tag{143}$$

We can now use these results to eliminate a' from Eq. (137) to find

$$\nabla\cdot\left[\left(1+\frac{\beta_D}{\beta_C}\frac{2\beta_C+C}{C}\right)\nabla p'+ko(\beta_D)\right]$$
$$+\frac{3\beta_D}{a_0^2}\frac{\omega^2}{\omega_0^2+2ib\omega-\omega^2}\,p'+\omega^4 o(\beta_D)=0. \tag{144}$$

For finite ω, consistency with the error terms forces one to drop the $O(\beta_D)$ corrections in the first term to find

$$\nabla^2 p'+\frac{3\beta_D}{a_0^2}\frac{\omega^2}{\omega_0^2+2ib\omega-\omega^2}\,p'=0, \tag{145}$$

with the corresponding well-known dispersion relation (see, Refs. 14–16)

$$\frac{k^2}{\omega^2}=\frac{3\beta_D}{a_0^2}\frac{1}{\omega_0^2+2ib\omega-\omega^2}, \tag{146}$$

from which one sees that $k=O(\omega\sqrt{\beta_D})$. The relative error here is $o(\beta_D)+\omega^2 o(1)$. In the limit $\omega\to 0$, however, Eq. (144) can be consistently approximated as

$$\nabla\cdot[(1+3\beta_D)\nabla p']+\frac{3\beta_D}{a_0^2}\frac{\omega^2}{\omega_0^2+2ib\omega}\,p'=0. \tag{147}$$

The corresponding dispersion relation is

$$\frac{k^2}{\omega^2}=\frac{3\beta_D}{a_0^2\omega_0^2}(1-3\beta_D). \tag{148}$$

Since, for vanishing frequency, $\omega_0\to(3p_{B0}/\rho a_0^2)^{1/2}$, to $O(\beta_D)$ this relation is equivalent to the well-known expression first given by Crespo[17]

$$\frac{\omega^2}{k^2}=\frac{p_{B0}}{\rho\beta_D}\frac{1+2\beta_D}{1-\beta_D}. \tag{149}$$

At low frequencies, but for wavelengths small compared with the characteristic scale for the variation of β_D, Eq. (147) admits approximate solutions in the form of slowly varying plane waves. Upon assuming $p'\propto\exp i(\omega t-\mathbf{k}\cdot\mathbf{x})$ and taking $C\simeq 1$, we find

$$(1+3\beta_D)\frac{k^2}{\omega^2}+3i\mathbf{k}\cdot\nabla\beta_D=\frac{\beta_D\rho}{p_{B0}}. \tag{150}$$

It is easy to show from this relation that $\mathrm{Im}\,k<0$ when the void fraction gradient makes an acute angle with the direction of propagation, while $\mathrm{Im}\,k>0$ in the opposite case. This effect may be understood by noting that waves propagating into a denser mixture need to excite more and more bubbles. Since the corresponding energy is extracted from the wave, its amplitude must decrease.

IX. COMPARISON WITH PREVIOUS WORK

Equation (145) valid for finite ω can be obtained starting from the lowest-order approximations to Eq. (132),

$$\rho_C\frac{\partial\mathbf{u}_C}{\partial t}+\nabla p_C=0, \tag{151}$$

and to the liquid continuity equation (135)

$$\nabla\cdot\mathbf{u}_C=\frac{\partial\beta_D}{\partial t}. \tag{152}$$

Indeed, with Eq. (134), the bubble number conservation equation (43) gives

$$\frac{\partial n}{\partial t}+3\nabla\cdot(n\mathbf{u}_C)=0. \tag{153}$$

With the small-particle approximation $\beta_D\simeq nv$ the right-hand side of Eq. (152) consists of two terms the ratio of which is, using Eqs. (153) and (127);

$$\frac{(4/3)\pi a^3\partial n/\partial t}{4\pi na^2\partial a/\partial t}\sim\frac{\nabla\cdot\mathbf{u}_C}{\partial a/\partial t}\sim\beta_D. \tag{154}$$

In keeping with the $O(\beta_D)$ accuracy of Eq. (151) and (152), the second term can therefore be neglected so that

$$\nabla\cdot\mathbf{u}_C=4\pi na^2\frac{\partial a}{\partial t}, \tag{155}$$

from which elimination of $\mathbf{u}_C$ gives Eq. (145).

In writing the previous equations we have restored the subscript C and purposely dropped the index zero and the prime used in the previous section as Eqs. (151) and (155) are also valid, to $O(\beta_D)$, in the case of nonlinear waves. The reason is that, as shown by Eq. (127) which does not depend on linearity, $\mathbf{u}_C=O(\beta_D)$ so that convective terms must be dropped from the equations for consistency. This fact was first pointed out by Caflisch *et al.*[18] who derived Eqs. (151) and (155) essentially by a multiple scales argument. Their method of analysis, however, gave no indication on the time dependence of $\mathbf{w}$ and n. Of course, for finite-amplitude waves, nonlinear effects are confined to the right-hand side of the continuity equation (155) and to the Rayleigh–Plesset equation. In a recent study,[8] this model was compared favorably with data for shock waves in dilute bubbly liquids. The previous equations are inappropriate to study the propagation of pressure waves in a *flowing* bubbly liquid, for which $\mathbf{u}_C$ is not small. The full equations presented in Secs. VI and VII must be used in this case.

An elegant theory for the linear problem beyond $O(\beta_D)$ accuracy has recently been given by Sangani.[19,20] An interesting result of this study is that, for finite ω, the leading-order correction to the previous $O(\beta_D)$ expressions is $O(\beta_D^{3/2})$. This result can be found in the present formalism by assuming, in the one-particle calculation, that the particle is surrounded by an effective compressible medium rather than by the pure liquid. With this assumption the analogue of the Rayleigh–Plesset equation (139) is the (linearized) Keller equation[21]

$$\rho a_0 \frac{\partial^2 a'}{\partial t^2} = \left(1 + \frac{a_0 k}{\omega}\right)\left(p_B' - p' - \frac{2\sigma}{a_0^2} a'\right) \tag{156}$$

which, following the same procedure leading to Eq. (146), gives

$$\frac{k^2}{\omega^2} = \frac{3\beta_D}{a_0^2} \frac{1}{\omega_0^2 + 2ib\omega - (1 - ika_0)\omega^2}. \tag{157}$$

Upon solving this relation approximately for k, Sangani's $O(\beta_D^{3/2})$ is recovered. Sangani also allowed for the deformation of the bubbles and pointed out the importance of two-bubble resonances in the dispersion relation.

Miksis and Ting[22] have recently presented a review of their earlier work[18,23,24] that includes bubble deformation and viscous effects. To deal with the finite (but small) volume fraction case, they use the method of homogeneization which does not seem suitable for a compressible two-phase continuum. For this reason their results are not in agreement with Sangani's.

In 1968 van Wijngaarden published a nonlinear model[25] for the flow of bubbly liquids. To compare our results with his, we add the two momentum equations (98) and (99) to find

$$\begin{aligned} \rho_C \beta_C \left[\frac{\partial \mathbf{u}_C}{\partial t} + (\mathbf{u}_C \cdot \nabla)\mathbf{u}_C\right] + \nabla p_C \\ = \rho_C \nabla \cdot \left\{\frac{1}{4} \beta_D [(\mathbf{w} - \mathbf{u}_C)\cdot(\mathbf{w} - \mathbf{u}_C) - \operatorname{Tr} \mathbf{M}_D]\mathbf{I} \right. \\ - \beta_D \left[\frac{5}{2} \dot{a}^2 + a\left(\frac{\partial \dot{a}}{\partial t} + \mathbf{w}\cdot\nabla \dot{a}\right)\right]\mathbf{I} - \frac{\beta_D}{2}(\mathbf{w} - \mathbf{u}_C) \\ \left. \times (\mathbf{w} - \mathbf{u}_C)\right\}. \end{aligned} \tag{158}$$

van Wijngaarden's momentum equation is obtained from this one by dropping the right-hand side. Evidently his model neglects therefore the relative motion, the dynamical effects of the bubble's compressibility, and the effect of the fluctuations in the bubbles' translational velocity. His radial equation is the same as our Eq. (101) without $\mathbf{M}_D$ and the term involving the relative velocity. He also assumed the bubbles to move with the liquid.

On the other hand, Eq. (158) coincides to $O(\beta_D)$ with that given in the later study by Biesheuvel and van Wijngaarden[26] except for the $\mathbf{M}_D$ term that they did not consider. In this work the single-bubble problem was solved by matching $\langle p_C \rangle_1$ at infinity to the mean *mixture* pressure, rather than the pure-liquid pressure as done here in the Appendix. This step might be justified by an "effective medium" type of approach and leads to a difference of $O(\beta_D)$ with the present results in Eq. (139). The bubble momentum equation of Ref. 26, on the other hand, has been obtained with a rather *ad hoc* procedure and approximates convective derivatives with partial time derivatives even for a flowing mixture.

In Ref. 1 we have given a detailed comparison of our model with that of Wallis[27] for the case of incompressible spheres finding equivalent results to first order in β_D. The additional terms due to compressibility of the disperse phase are also very readily seen to agree exactly to first order in β_D. To the extent that the results of Drew and collaborators also are in agreement with those of Wallis,[28] our model also conforms to Drew's. However, in our derivation, we did not have to rely on tools only applicable to potential flow nor to have recourse to *ad hoc* cell models.

X. CONCLUSIONS

We have studied the problem of the flow of a mixture of spherical bubbles and an inviscid liquid by means of the ensemble averaging techniques introduced in Ref. 1. We have thus been led naturally to a model of the two-fluid type. A closure correct to first-order in the gas volume fraction β_D has been effected explicitly with no need for *ad hoc* approximations. On the basis of the resulting dilute-limit model we have examined the propagation of linear pressure waves in a bubbly liquid.

The method that we have used is systematic and has enabled us to obtain results of greater generality and with greater level of detail than previously possible. For instance, we include the effect of fluctuations in the bubble velocity and in the evolution of their radius. While the modeling of these effects may be difficult, it is conceptually significant that they readily arise in our approach. It may also be noted that our general method is not restricted to potential flow. Examples of its application to Stokes flow, convection-diffusion, and thermocapillary motion can be found in Ref. 29 while an analysis of the particle stress is given in Ref. 2. Several other problems are currently under investigation.

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APPENDIX: CLOSURE IN DILUTE LIMIT

We now briefly describe the dilute-limit closure of Sec. V. The calculation follows that given in Appendix B of Ref. 1 with only slight modifications. In order to avoid duplication, we refer the reader to that work and to Ref. 29 for details.

We start by changing the frame of reference to a system in which the particle centered at y is at rest and indicate with primes quantities evaluated in this frame. With $\langle \mathbf{u}_C' \rangle = \nabla \phi'$ and $\langle \mathbf{u}_C' \rangle_1 = \nabla \phi_1'$ we then have, as in Ref. 1,

$$\frac{\langle p_C \rangle_1 - \langle p_C \rangle}{\rho_C} = \frac{\partial}{\partial t}(\phi' - \phi_1') + \frac{1}{2}(\langle \mathbf{u}_C' \rangle \langle \mathbf{u}_C' \rangle - \langle \mathbf{u}_C' \rangle_1 \cdot \langle \mathbf{u}_1' \rangle_1). \tag{A1}$$

As in Ref. 1, we expand ϕ' around the point $\mathbf{y}$ by writing

$$\phi'(\mathbf{z})=\Psi_0+r\Psi_1+r^2\Psi_2+\cdots, \tag{A2}$$

where

$$\Psi_0=\phi'(\mathbf{y}), \quad \Psi_1=\frac{(z-y)_i}{r}\,\partial_i\phi'(\mathbf{y}),$$
$$\Psi_2=\frac{(z-y)_i(z-y)_j}{2r^2}\,\partial_i\partial_j\phi'(\mathbf{y}). \tag{A3}$$

If a sphere of radius a expanding with radial velocity $\dot{a}$ is now placed at $\mathbf{y}$, the velocity potential becomes

$$\phi_1'=-\frac{\dot{a}(t)a^2(t)}{r}+\Psi_0+r\left[1+\frac{1}{2}\left(\frac{a}{r}\right)^3\right]\Psi_1+r^2\times\left[1+\frac{2}{3}\left(\frac{a}{r}\right)^5\right]\Psi_2+\cdots. \tag{A4}$$

From this expression the component $\langle u_{Ci}'\rangle_1$ of the velocity $\langle \mathbf{u}_C'\rangle_1$ at $\mathbf{z}$ in the i direction is readily found as

$$\begin{aligned}\langle u_{Ci}'\rangle_1(\mathbf{z})=&\langle u_{Ci}'\rangle(\mathbf{z})+\frac{\dot{a}a^2}{r^2}\frac{z_i-y_i}{r}+\frac{1}{2}\left(\frac{a}{r}\right)^3\\&\times\left[\delta_{ik}-3\,\frac{(z-y)_i(z-y)_k}{r^2}\right]\langle u_{Ck}'\rangle(\mathbf{y})\\&+\frac{1}{3}\left(\frac{a}{r}\right)^5\left[2\delta_{ik}-5\,\frac{(z-y)_i(z-y)_k}{r^2}\right]\\&\times(z-y)_m\partial_m\langle u_{Ck}'\rangle(\mathbf{y})+\cdots,\end{aligned} \tag{A5}$$

where

$$\begin{aligned}\langle u_{Ci}'\rangle(\mathbf{z})&=\partial_i\phi'(\mathbf{y})+(z-y)_j\partial_i\partial_j\phi'(\mathbf{y})+\cdots\\&=\langle u_{Ci}'\rangle(\mathbf{y})+(z-y)_j\partial_j\langle u_{Ci}'\rangle+\cdots.\end{aligned} \tag{A6}$$

With these results, we can write the pressure distribution on the surface of the particle at $\mathbf{y}$ as

$$\begin{aligned}\langle p_C\rangle_1-\langle p_C\rangle=\rho_C\Big[&\ddot{a}+\frac{3}{2}\dot{a}^2-\frac{3}{2}\dot{a}n_iu_i'\\&-\frac{1}{2}an_i\frac{\partial u_i'}{\partial t}-\frac{1}{8}(5u_k'u_k'-9n_kn_ju_k'u_j')\\&-\frac{1}{2}a(3n_iu_k'\partial_ku_i'-5n_in_jn_ku_j'\partial_iu_k')\Big]\\&+O(a^2).\end{aligned} \tag{A7}$$

Upon substitution into expression (61) of $\mathbf{A}_a$ one finds

$$\mathbf{A}_a=-\frac{3}{2a}\rho_C(\overline{\dot{a}\mathbf{w}}^{\,a}-\bar{\dot{a}}^a\langle\mathbf{u}_C\rangle)-\frac{1}{2}\rho_C\left[\bar{\dot{\mathbf{w}}}^a-\frac{\partial\langle\mathbf{u}_C\rangle}{\partial t}-\langle\mathbf{u}_C\rangle\cdot\nabla\langle\mathbf{u}_C\rangle\right]. \tag{A8}$$

The term $\bar{\dot{\mathbf{w}}}^a$ can be expressed by means of Eq. (31) and the product $\overline{\dot{a}\mathbf{w}}^{\,a}$ broken up introducing the correlation N_a defined by Eq. (79). Furthermore, noting that $3v_a/a=dv_a/da$, we can write Eq. (A8) as

$$\begin{aligned}\beta_a\mathbf{A}_a=&\frac{1}{2}\rho_C\beta_a\left\{\frac{\partial\langle\mathbf{u}_C\rangle}{\partial t}+\langle\mathbf{u}_C\rangle\cdot\nabla\langle\mathbf{u}_C\rangle-\frac{\partial\bar{\mathbf{w}}^a}{\partial t}-\bar{\mathbf{w}}^a\cdot\nabla\bar{\mathbf{w}}^a\right\}\\&+\frac{1}{2}\rho_Cn_a\bar{\dot{a}}^a\frac{\partial}{\partial a}[v_a(\langle\mathbf{u}_C\rangle-\bar{\mathbf{w}})]+\frac{1}{2}\rho_C\nabla\cdot(\beta_a\mathbf{M}_a)\\&+\frac{1}{2}\rho_C\frac{\partial}{\partial a}(\beta_a\mathbf{N}_a).\end{aligned} \tag{A9}$$

The calculation of $\mathbf{T}_a$ according to Eq. (64) is similar. The integrand only differs from that of $\mathbf{A}_a$ by the presence of an extra factor $\mathbf{n}$, which singles out different terms in the integrand. We define $\mathbf{T}_a'$ by

$$\beta_a\mathbf{T}_a=\int d^4p\;P(1;t)\mathbf{T}_a', \tag{A10}$$

and find

$$\begin{aligned}\mathbf{T}_a'\equiv&-\int_{|\mathbf{x}-\mathbf{y}|=a}a\mathbf{nn}(\langle p_C\rangle_1-\langle p_C\rangle)dS_y\\=&-\rho_Cv_a(\ddot{a}a+\tfrac{3}{2}\dot{a}^2)\mathbf{I}+\rho_Cv_a\left[\frac{2}{5}(\mathbf{w}-\langle\mathbf{u}_C\rangle)\cdot(\mathbf{w}-\langle\mathbf{u}_C\rangle)\right.\\&\left.\times\mathbf{I}-\frac{9}{20}(\mathbf{w}-\langle\mathbf{u}_C\rangle)(\mathbf{w}-\langle\mathbf{u}_C\rangle)\right],\end{aligned} \tag{A11}$$

from which the result (80) given before is readily obtained.

As a check of these results, the quantity Q_a defined by Eq. (65) can also be calculated directly and compared with the trace of $\mathbf{T}_a$ according to Eq. (66).

The last quantity that remains to be calculated is the Reynolds stress (39), which in the present dilute limit can be approximated as

$$\begin{aligned}\mathbf{M}_C(\mathbf{x},t)\simeq&-\langle(\langle\mathbf{u}_C\rangle_1-\langle\mathbf{u}_C\rangle)(\langle\mathbf{u}_C\rangle_1-\langle\mathbf{u}_C\rangle)\rangle_1\\=&-\frac{1}{\beta_C}\int da\int d^4p\int\beta_C^1P(\mathbf{y},a,\mathbf{p};t)\\&\times(\langle\mathbf{u}_C\rangle_1-\langle\mathbf{u}_C\rangle)(\langle\mathbf{u}_C\rangle_1-\langle\mathbf{u}_C\rangle)d^3y.\end{aligned} \tag{A12}$$

From the definition (14) it follows that

$$\begin{aligned}\beta_C^1(\mathbf{x},t|1)=&1-H(a-|\mathbf{x}-\mathbf{y}|)-\int da^{(2)}\int_{|\mathbf{x}-\mathbf{y}^{(2)}|\le a}a^3y^{(2)}\\&\times\int d^4p^{(2)}\,P(1,2;t)/P(1;t),\end{aligned} \tag{A13}$$

so that, for $|\mathbf{x}-\mathbf{y}|<a$, $\beta_C^1=0$. Furthermore, in the dilute limit, $\beta_C^1\simeq\beta_C$ for $|\mathbf{x}-\mathbf{y}|>a$ so that

$$\begin{aligned}\mathbf{M}_C(\mathbf{x},t)\simeq&-\int da\int d^4p\int_{|\mathbf{x}-\mathbf{y}|\ge a}P(\mathbf{y},a,\mathbf{p};t)\\&\times(\langle\mathbf{u}_C\rangle_1-\langle\mathbf{u}_C\rangle)(\langle\mathbf{u}_C\rangle_1-\langle\mathbf{u}_C\rangle)d^3y.\end{aligned} \tag{A14}$$

After substitution of the expression (A5), one finds that each term of the volume integral has the form

$$\mathbf{I}=\int_{|\mathbf{r}|\ge a}g(a/r,\mathbf{n})f(\mathbf{x},\mathbf{r},t)d^3r, \tag{A15}$$

where $f(\mathbf{x},\mathbf{r},t)$ is the product of the undisturbed velocity field and the probability distribution function $P(\mathbf{y},a,\mathbf{w},\dot{a};t)$, and $g(a/r,\mathbf{n})$ decays with increasing r. After adding and subtracting $f(\mathbf{x},0,t)$, the integral can be written as

$$\mathbf{I}=\int_{|\mathbf{r}|\geqslant a} g(a/r,\mathbf{n})f(\mathbf{x},0,t)d^3r+\int_{|\mathbf{r}|\geqslant a} g(a/r,\mathbf{n}) \times[f(\mathbf{x},\mathbf{r},t)-f(\mathbf{x},0,t)]d^3r. \tag{A16}$$

After the change of variable $\mathbf{r}=a\mathbf{s}$, the second term can be written as

$$\mathbf{R}=a^3\int_{|s|\geqslant 1} g(1/s,\mathbf{n})[f(\mathbf{x},a\mathbf{s},t)-f(\mathbf{x},\mathbf{0},t)]d^3s. \tag{A17}$$

If this integral exists, as we assume, it is a function of a vanishing as a tends to zero. Hence we conclude that $\mathbf{R}=o(a^3)$ for small a, and can be neglected in the present dilute limit. Therefore, the volume integral in Eq. (A14) can be calculated approximately by using $\langle u_{Ck}\rangle(\mathbf{x},t)$ in place of $\langle u_{Ck}\rangle(\mathbf{y},t)$ and $P(\mathbf{x},a,\mathbf{w},\dot{a};t)$ instead of $P(\mathbf{y},a,\mathbf{w},\dot{a};t)$. This simplifies the calculation. With the definition

$$\mathbf{M}'_C=-\int_{|\mathbf{x}-\mathbf{y}|\geqslant a} d^3y(\langle\mathbf{u}_C\rangle_1-\langle\mathbf{u}_C\rangle)(\langle\mathbf{u}_C\rangle_1-\langle\mathbf{u}_C\rangle), \tag{A18}$$

we have

$$\mathbf{M}'_C=-v_a\left[\dot{a}^2+\frac{3}{20}(\mathbf{w}-\langle\mathbf{u}_C\rangle)\cdot(\mathbf{w}-\langle\mathbf{u}_C\rangle)\right]\mathbf{I} +\frac{1}{20}v_a(\mathbf{w}-\langle\mathbf{u}_C\rangle)(\mathbf{w}-\langle\mathbf{u}_C\rangle). \tag{A19}$$

Finally, to find $\mathbf{M}_C$, we integrate over $\mathbf{w}$, $\dot{a}$, and a,

$$\mathbf{M}_C=\int da\int d\dot{a}\int d^3w\ P(\mathbf{x},a,\mathbf{w},\dot{a};t)M'_C, \tag{A20}$$

which gives expression (82) quoted earlier. With the neglect of the particle velocity fluctuations, this result coincides with that found by Biesheuvel and van Wijngaarden[24] by volume averaging.

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